Journal of Computational and Applied Mathematics 236 (2012) 3052–3057

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A note on the equilibria of an economic model with local competition "à la Cournot"

Juan L.G. Guirao [a,*], Miguel A. López [b], Jaume Llibre [c], Raquel Martínez [b]

[a] Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, Hospital de Marina, 30203-Cartagena, Región de Murcia, Spain
[b] Departamento de Matemáticas, Universidad de Castilla-La Mancha, Campus de Cuenca, 16071-Cuenca, Castilla-La Mancha, Spain
[c] Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra, 08193-Barcelona, Catalonia, Spain

ARTICLE INFO

MSC:
37N40

Keywords:
Discrete dynamical systems and economics
Cournot duopoly
Cournot-like model
Equilibrium point

ABSTRACT

Guirao and Rubio (2010) [12] introduce an economic model, which generalizes the classical duopoly of Cournot type, where the competitors are located around a circle or a line and each firm competes "à la Cournot" with its right and left neighboring. For the case of having three and four players, we describe completely the bifurcations of equilibria in terms of the production costs of each firm and we study the stability of them. Moreover, for the case of four players we provide some information on the two-periodic orbits of the system.

1. Introduction and statement of the main result

Cournot duopoly was introduced by Cournot [1] who is considered one of the forerunners of the modern microeconomics. The process consists of two firms which produce an identical good and which compete for the market. In each step of the process the firms decide the amount of product to be introduced in the market, and for making this decision both firms should know the amount of product introduced in the market in the previous step by the rival firm. This economic process is mathematically modeled by the following two-dimensional discrete dynamical system

$$F(x, y) = (g(y), f(x)) \tag{1}$$

where f, g are continuous self-maps defined on a compact interval which can be considered, without loss of generality, by normalization $[0, 1]$. The maps f and g are called the *reaction functions* and determine the decisions made by the firms.

Note that if firm A put on the market at the beginning of the game α_0 product and firm B put β_0, in the next step of the game firm A will produce $g(\beta_0)$, i.e. an amount of product which directly depends on the production level of the firm B in the previous step, on the other hand, firm B will produce $f(\alpha_0)$ and so on. Therefore, whole of the process is governed by the dynamics of the discrete system (1) which strongly depends on the dynamics of the one-dimensional interval maps f and g.

Duopoly is an intermediate situation between monopoly and perfect competition, and analytically is a more complicated case. The reason for this is that oligopolist must consider not only the behaviors of the customers, but also those of the competitors and their reactions. Thus this model has been studied in the literature from different points of view; see for instance [2–9] or [10].

While dynamic properties of duopolies have been extensively studied, adjustment dynamics in Cournot processes with more than two players have received much less attention as a consequence of the difficulties which appear for studying discrete dynamical systems with a dimension higher than two. The direct generalization of the Cournot duopoly situation is

* Corresponding author.
E-mail addresses: juan.garcia@upct.es (J.L.G. Guirao), mangel.lopez@uclm.es (M.A. López), jllibre@mat.uab.cat (J. Llibre), raquel.martinez@uclm.es (R. Martínez).

0377-0427/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2010.11.030

the Cournot oligopoly, i.e. consider n firms which produce an identical good and in each step of the process any firm knows the amount of product generated by the $n-1$ rival firms in the previous step. Now, the systems which model the situation are given by

$$F(x_1, x_2, \dots, x_n) = (f_1(x_2, x_3, \dots, x_n), f_2(x_1, x_3, \dots, x_n), \dots, f_n(x_1, x_2, \dots, x_{n-1})) \tag{2}$$

where $f_i : [0, 1]^{n-1} \to [0, 1]$ is a continuous map. We note that the reaction function f_i depends on $n-1$ variables of indices $j \in \{1, 2, \dots, n\}, j \neq i$.

To study the dynamics for a system like (2) is quite complicated by the ignorance of the topological dynamics of n-dimensional discrete dynamical systems with $n > 2$ (e.g., note that the possible ω-limit sets of the orbits for these type of systems are not characterized).

Thus, if we want to have some chance of describing dynamics we need to simplify the system with the cost of losing information by the players on the production level of the rivals.

In [11] the following model called Cournot-like system is introduced.

Definition 1. A continuous map ϕ from $[0, 1]^n$ into itself is Cournot-like if it is of the form

$$\phi(x_1, x_2, \dots, x_n) = (\phi_{\sigma(1)}(x_{\sigma(1)}), \dots, \phi_{\sigma(n-1)}(x_{\sigma(n-1)}), \phi_{\sigma(n)}(x_{\sigma(n)})),$$

where $\phi_i : [0, 1] \to [0, 1]$ is continuous, $i \in \{1, 2, \dots, n\}$ and σ is a cyclic permutation of the set $\{1, 2, \dots, n\}$.

In the economic situation models by these type of systems the level of information is quite limited because any player firm only has information on the production level of one of the other firms in the previous step of the process. For these type of systems, see [11], there is a characterization of the dynamical simplicity.

From our point of view Cournot-like models do not represent a truthful economic situation since it is very difficult to explain the fact that each player firm can only have information on other firm having a complete ignorance on the rest of player behavior. For that reason Guirao and Rubio [12] introduce a new model where the information level is higher than in Cournot-like ones and where there is more chance for describing dynamical properties. See next section for a concrete description of this model. The aim of the present paper is, for dimensions 3 and 4, to describe completely the bifurcations of equilibria of this new model in terms of the production costs of each firm and we study the stability of them. Moreover, for the case of four players we are able to prove that the system has no two-periodic orbits. The statement of our main results is the following.

Theorem 2. *Let (F_3, Ω_3) be the discrete dynamical system introduced in (3). Then*

(i) *if c_1, c_2 and c_3 are not equal, there exists a unique equilibrium equal to $(0, 0, 0)$ which is strongly unstable;*
(ii) *if $c_i = c$, $i = 1, 2, 3$, there exists another equilibrium, apart from that stated in (i), of the form $(\frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c})$ which is locally an attractor.*

Theorem 3. *Let (F_4, Ω_4) be the discrete dynamical system introduced in (4). Then*

(i) *if c_1, c_2, c_3 and c_4 are arbitrary positive constants, the point $(0, 0, 0, 0)$ is a strongly unstable equilibrium;*
(ii) *if $c_i = c$, $i = 1, 2, 3, 4$, the point $(\frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c})$ is a stable equilibrium;*
(iii) *if $c_1 = c_3$ and $c_2 \neq c_4$ the point $\left(\alpha^2, \sqrt{\frac{2\alpha^2}{c_2}} - 2\alpha^2, \alpha^2, \sqrt{\frac{2\alpha^2}{c_4}} - 2\alpha^2\right)$, where α is the unique root of the cubic polynomial*

$$-\sqrt{2c_2c_4}(\sqrt{c_2} + \sqrt{c_4}) + 2(c_1c_2 + c_1c_4 + 2c_2c_4 + 2c_1\sqrt{c_2c_4})X - 6c_1\sqrt{2c_2c_4}(\sqrt{c_2} + \sqrt{c_4})X^2 + 9c_1c_2c_4X^3,$$

is an equilibrium. Its stability depends on the values of c_2 and c_4. In the case $c_i = c$, $i = 1, 2, 3, 4$, the equilibrium coincides with the presented one in (ii);
(iv) *if $c_2 = c_4$ and $c_1 \neq c_3$, the equilibria are symmetric to the ones of case (iii) interchanging z with w, x with y, c_2 with c_1, and c_4 with c_3.*

Moreover, the system has no two-periodic orbits.

2. The model

Let $N = \{1, 2, \dots, n\}$ be the set of players (i.e., rival firms which produces an identical good) and assume that they are physically located around a circle or a line. We assume that the firms compete "à la Cournot" in a local way, i.e., each firm $i \in N$ compete with its closest neighboring in the right and left direction. Let $B_i^\alpha \subset N$ be the neighboring located at a distance equal to α of the firm i in the right and left direction. If we denote by $(x_1, \dots, x_n)$ the production of the firms in some moment and by $(c_1, c_2, \dots, c_n)$ their production costs, the best response function for the firm i will have the form

$$\phi_i(x_{B_i^\alpha}) = \sqrt{\frac{\sum_{k\in B_i^\alpha} x_k}{c_i}} - \sum_{k\in B_i^\alpha} x_k.$$

We consider that B_i^α is composed by the left and right neighboring, i.e., $\alpha = 1$. In this case, we note that if $n = 2$, we have the classical situation of the Cournot duopoly.

If we suppose that the number of players is equal to three, the model is governed by the three-dimensional discrete dynamical system given by

$$F_3(x, y, z) = \left(\sqrt{\frac{y+z}{c_1}} - (y+z), \sqrt{\frac{x+z}{c_2}} - (x+z), \sqrt{\frac{x+y}{c_3}} - (x+y)\right) \tag{3}$$

defined in $\Omega_3 = \{(x, y, z) \in \mathbb{R}^3 : x + y > 0, x + z > 0, y + z > 0\}$.

In the same way, for the case of having four firms, the model is defined by a four-dimensional discrete system $F_4(x, y, z, w)$ of the form (x, y, z, w) goes to

$$\left(\sqrt{\frac{y+w}{c_1}} - (y+w), \sqrt{\frac{x+z}{c_2}} - (x+z), \sqrt{\frac{y+w}{c_3}} - (y+w), \sqrt{\frac{x+z}{c_4}} - (x+z)\right) \tag{4}$$

defined in $\Omega_4 = \{(x, y, z, w) \in \mathbb{R}^4 : y + w > 0, x + z > 0\}$.

The objective of the next section is to study the equilibria of systems (3) and (4) in terms of production costs $c_i's$. In the case $n = 4$ we give some information on the two-periodic orbits.

3. Proof of Theorem 2

In this section we state the equilibria and their bifurcations for the system (3) depending on $c_i's$.

Proof of Theorem 2. We consider the map F_3 given by

$$(x, y, z) \to \left(\sqrt{\frac{y+z}{c_1}} - (y+z), \sqrt{\frac{x+z}{c_2}} - (x+z), \sqrt{\frac{x+y}{c_3}} - (x+y)\right).$$

A point (x, y, z) will be an equilibrium point if and only if it is held simultaneously

- $k_1 = \sqrt{\frac{y+z}{c_1}} - (x + y + z) = 0,$
- $k_2 = \sqrt{\frac{x+z}{c_2}} - (x + y + z) = 0,$
- $k_3 = \sqrt{\frac{x+y}{c_3}} - (x + y + z) = 0.$

Clearly, one solution of the system, for any value of $c_i's$, is $(0, 0, 0)$. Now, consider the equivalent system $e_1 = e_2 = e_3 = 0$ where

- $e_1 = k_1 - k_2$
- $e_2 = k_1 - k_3$
- $e_3 = k_2 - k_3$.

It is easy to note that the case $c_3 = c_1 + c_2$ does not provide any fixed point different from the origin. If c_3 is different from $c_1 + c_2$, the solution of the system $e_1 = e_2 = e_3 = 0$ is

$$x = -\frac{2c_1 z}{c_1 + c_2 - c_3} + \sqrt{\frac{2z}{c_1 + c_2 - c_3}},$$

$$y = -\frac{2c_2 z}{c_1 + c_2 - c_3} + \sqrt{\frac{2z}{c_1 + c_2 - c_3}},$$

$$z = -\frac{2c_3 z}{c_1 + c_2 - c_3} + \sqrt{\frac{2z}{c_1 + c_2 - c_3}}.$$

This system has a solution different from $z = 0$ in the case $c_i = c, i = 1, 2, 3$. Thus, the solution has the form

$$\left(-2z + \sqrt{\frac{2z}{c}}, -2z + \sqrt{\frac{2z}{c}}, -2z + \sqrt{\frac{2z}{c}}\right),$$

now for the equilibrium condition

$$-2z + \sqrt{\frac{2z}{c}} = z,$$

and therefore the equilibrium point has the form $\left(\frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c}\right)$.

In short, we obtain that $(0, 0, 0)$ is always an equilibrium point, and when $c_1 = c_2 = c_3$ then $\left(\frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c}\right)$ is another equilibrium point.

For studying the stability of the equilibria we compute the matrix M composed by the partial derivatives of the reaction functions of F. Indeed,

$$M = \begin{pmatrix} \frac{\partial F_3^1}{\partial x} & \frac{\partial F_3^1}{\partial y} & \frac{\partial F_3^1}{\partial z} \\ \frac{\partial F_3^2}{\partial x} & \frac{\partial F_3^2}{\partial y} & \frac{\partial F_3^2}{\partial z} \\ \frac{\partial F_3^3}{\partial x} & \frac{\partial F_3^3}{\partial y} & \frac{\partial F_3^3}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & -1 + \frac{1}{2c_1\sqrt{\frac{y+z}{c_1}}} & -1 + \frac{1}{2c_1\sqrt{\frac{y+z}{c_1}}} \\ -1 + \frac{1}{2c_2\sqrt{\frac{y+z}{c_2}}} & 0 & -1 + \frac{1}{2c_2\sqrt{\frac{y+z}{c_2}}} \\ -1 + \frac{1}{2c_3\sqrt{\frac{y+z}{c_3}}} & -1 + \frac{1}{2c_3\sqrt{\frac{y+z}{c_3}}} & 0 \end{pmatrix}.$$

In this setting, we conclude that the origin is strongly unstable in all directions because the partial derivatives which appear in the matrix M tend to infinity when $(x, y, z) \to 0$. Let us assume that the costs $c_i's$ are equal to c, then the characteristic polynomial of the matrix M evaluated at the equilibrium point $\left(\frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c}\right)$ is equal to $P_\lambda = \frac{1}{32}(-1 + 6\lambda - 32\lambda^3)$. Now note that the roots of P_λ are equal to $\frac{1}{4}$ double and $-\frac{1}{2}$. Thus, the equilibrium point $\left(\frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c}\right)$ is locally an attractor because the absolute value of the eigenvalues at this fixed point are smaller than 1, ending the proof. □

Remark 4. The dynamics on the straight line $x = y = z$ when $c_1 = c_2 = c_3 = c$, is equivalent to the dynamics of an one-dimensional map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{\frac{2x}{c}} - 2x$. On the line $x = y = z$ the fixed point $x = \frac{2}{9c}$ is an attractor. The point $x = 0$ is strongly unstable.

4. Proof of Theorem 3

The equations of the equilibria points of the system (Ω_4, F_4), where F_4 is the transformation such that every (x, y, z, w) go to

$$\left(\sqrt{\frac{y+w}{c_1}} - (y+w), \sqrt{\frac{x+z}{c_2}} - (x+z), \sqrt{\frac{y+w}{c_3}} - (y+w), \sqrt{\frac{x+z}{c_4}} - (x+z)\right)$$

are the following:

- $k_1 = \sqrt{\frac{y+w}{c_1}} - (x + y + w) = 0,$
- $k_2 = \sqrt{\frac{x+z}{c_2}} - (x + y + z) = 0,$
- $k_3 = \sqrt{\frac{y+w}{c_3}} - (y + w + z) = 0,$
- $k_4 = \sqrt{\frac{x+z}{c_4}} - (x + z + w) = 0.$

On the other hand the matrix M composed by the partial derivatives of the reaction functions of F_4 is

$$M = \begin{pmatrix} 0 & -1 + \frac{1}{2c_1\sqrt{\frac{w+y}{c_1}}} & 0 & -1 + \frac{1}{2c_1\sqrt{\frac{w+y}{c_1}}} \\ -1 + \frac{1}{2c_2\sqrt{\frac{x+z}{c_2}}} & 0 & -1 + \frac{1}{2c_2\sqrt{\frac{x+z}{c_2}}} & 0 \\ 0 & -1 + \frac{1}{2c_3\sqrt{\frac{w+y}{c_3}}} & 0 & -1 + \frac{1}{2c_3\sqrt{\frac{w+y}{c_3}}} \\ -1 + \frac{1}{2c_4\sqrt{\frac{x+z}{c_4}}} & 0 & -1 + \frac{1}{2c_4\sqrt{\frac{x+z}{c_4}}} & 0 \end{pmatrix}$$

where entry $a_{i,j}$ correspond to $\frac{\partial F_4^i}{\partial \alpha_j}$ with $\alpha_1 = x, \alpha_2 = y, \alpha_3 = z$ and $\alpha_4 = w$.

Clearly, by the two previous expressions, $(0, 0, 0, 0)$ is always a fixed point that is strongly unstable because the derivatives of the linear part of the map at this point go to infinity.

Now we consider three different cases.

- Case $c_1 = c_2 = c_3 = c$. In this case the equilibria equations are reduced to

$$-x - 2y + \sqrt{\frac{2y}{c}} = 0, \qquad -2x + \sqrt{\frac{2x}{c}} - y = 0.$$

Solving this system we obtain that there is an additional equilibrium point of the form $(\frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c})$ which is locally stable because its eigenvalues are $(0, 0, \frac{-1}{2}, \frac{1}{2})$ if we assume that $c > 0$.

- Case $c_3 = c_1$ and $c_4 \neq c_2$. In this setting equations of equilibria have the form

$$-w - x - y + \sqrt{\frac{x+y}{c_1}} = 0 \quad \text{where } w = -2x + \sqrt{\frac{2x}{c_4}} \text{ and } y = -2x + \sqrt{\frac{2x}{c_2}}.$$

Computing, we reduce the problem to solve

$$3x - \sqrt{\frac{2x}{c_2}} - \sqrt{\frac{2x}{c_4}} + \sqrt{\frac{-4x + \sqrt{\frac{2x}{c_2}} + \sqrt{\frac{2x}{c_4}}}{c_1}} = 0. \tag{5}$$

We can assume that x is not zero, otherwise we obtain the equilibrium point $(0, 0, 0, 0)$. Dividing Eq. (5) by $\sqrt{x}$ we obtain

$$3\sqrt{x} - \sqrt{\frac{2}{c_2}} - \sqrt{\frac{2}{c_4}} + \sqrt{\frac{-4 + \sqrt{\frac{2}{c_2 x}} + \sqrt{\frac{2}{c_4 x}}}{c_1}} = 0. \tag{6}$$

Replacing variable $\sqrt{x}$ by X in Eq. (6), we have

$$3X - \sqrt{\frac{2}{c_2}} - \sqrt{\frac{2}{c_4}} + \sqrt{\frac{-4X + \sqrt{\frac{2}{c_2}} + \sqrt{\frac{2}{c_4}}}{c_1 X}} = 0. \tag{7}$$

Now, we eliminate the squareroot containing the variable X in Eq. (7) taking squares, but note that doing this we can add fictious solutions. Thus, we have

$$-\sqrt{2c_2c_4}(\sqrt{c_2} + \sqrt{c_4}) + 2(c_1c_2 + c_1c_4 + 2c_2c_4 + 2c_1\sqrt{c_2c_4})X \tag{8}$$

$$-6c_1\sqrt{2c_2c_4}(\sqrt{c_2} + \sqrt{c_4})X^2 + 9c_1c_2c_4X^3 = 0, \tag{9}$$

and therefore computing is stated that the point

$$\left(\alpha^2, \sqrt{\frac{2\alpha^2}{c_2}} - 2\alpha^2, \alpha^2, \sqrt{\frac{2\alpha^2}{c_4}} - 2\alpha^2\right),$$

where α is the unique root of the cubic polynomial (8) is an equilibrium of the system different from the origin. Note that the stability of this equilibrium depends on the values of c_2 and c_4 and in the case $c_i = c, i = 1, 2, 3, 4$, the equilibrium coincides with the presented one in (ii).

- Case $c_4 = c_2$ and $c_3 \neq c_1$. In fact the equilibria points are symmetric to ones obtained in the previous case interchanging z with w, x with y, c_2 with c_1, and c_4 with c_3.

For the case $c_i = c, i = 1, 2, 3, 4$, we study the two-periodic orbits of the system (i.e., non-equilibria points with the property $F^2(x, y, z, w) = (x, y, z, w)$.) The equations of the two-periodic points, for the case $c_i = c, i = 1, 2, 3, 4$, are the following.

- $g_1 = x + 2z - 2\sqrt{\frac{x+z}{c}} + \sqrt{\frac{-2x-2z+2\sqrt{\frac{x+z}{c}}}{c}} = 0,$
- $g_2 = 2w + y - 2\sqrt{\frac{w+y}{c}} + \sqrt{\frac{-2w-2y+2\sqrt{\frac{w+y}{c}}}{c}} = 0,$
- $g_3 = 2x + z - 2\sqrt{\frac{x+z}{c}} + \sqrt{\frac{-2x-2z+2\sqrt{\frac{x+z}{c}}}{c}} = 0,$
- $g_4 = w + 2y - 2\sqrt{\frac{w+y}{c}} + \sqrt{\frac{-2w-2y+2\sqrt{\frac{w+y}{c}}}{c}} = 0.$

In this setting, $g_1 - g_3 = -x + z$ and $g_2 - g_4 = w - y$ being $z = x$ and $w = y$. Therefore, the unique solution of this system is the fixed point $(\frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c})$, ending the proof. □

Acknowledgements

The authors thank the valuable help of Ricardo Martínez from the Economics Department of the Brown University.

This work was partially supported by MCYT/FEDER grant number MTM2008-03679/MTM, Fundación Séneca de la Región de Murcia, grant number 08667/PI/08 and JCCM (Junta de Comunidades de Castilla-La Mancha), grant number PEII09-0220-0222. The second author was partially supported by MCYT/FEDER grant number MTM2008-03437, by CICYT grant number 2009SGR 410 and ICREA Academia.

References

[1] A. Cournot, Réserches sur les principles mathemátiques de la théorie de la richesse, Hachette, Paris, 1938.
[2] G.I. Bischi, L. Gardini, C. Mammana, Multistability and cyclic attractors in duopoly games, Chaos Solitions Fractals 11 (2000) 543–564.
[3] R.A. Dana, L. Montrucchio, Dynamical complexity in duopoly games, J. Econom. Theory 40 (1986) 40–56.
[4] W. Govaerts, R. Khoshsiar Ghaziani, Stable cycles in Cournot duopoly model of Kopel, J. Comput. Appl. Math. 218 (2008) 247–258.
[5] M. Kopel, Simple and complex adjustement dynamics in Cournot duopoly games, Chaos Solitions Fractals 7 (1996) 2031–2048.
[6] T. Puu, Chaos in duopoly pricing, Chaos Solitions Fractals 1 (1991) 573–581.
[7] T. Puu, Nonlinear Economic Dynamics, Springer-Verlag, Berlin, 1997.
[8] D. Rand, Exotic phenomena in games and duopoly models, J. Math. Econom. 5 (1978) 173–184.
[9] M.S. Santos, J. Vigo-Aguiar, Error bounds for a numerical solution for dynamic economic models, Appl. Math. Lett. 9 (4) (1996) 41–45.
[10] M.S. Santos, J. Vigo-Aguiar, Analysis of a numerical dynamic programming algorithm applied to economic models, Econometrica 60 (2) (1998) 409–426.
[11] J.L.G. Guirao, R.G. Rubio, Detecting simple dynamics in Cournot-like models, J. Comput. Appl. Math. 233 (4) (2009) 1091–1095.
[12] J.L.G. Guirao, R.G. Rubio, Extensions of Cournot duopoly: an applied mathematical view, Appl. Math. Lett. 23 (2010) 836–838.